On Gibbsianness of Random Fields

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Abstract

The problem of characterization of Gibbs random fields is considered. Various Gibbsianness criteria are obtained using the earlier developed one-point framework which in particular allows to describe random fields by means of either one-point conditional or one-point finite-conditional distributions. The main outcome are the criteria in terms of one-point finite-conditional distribution, one of which can be taken as a purely probabilistic definition of Gibbs random field.

Résumé

Le problème considéré est celui de la caractérisation des champs aléatoires de Gibbs. Divers critères de gibbsianité sont obtenus en utilisant l'approche uniponctuelle développée antérieurement et qui permet en particulier de décrire les champs aléatoires soit par des probabilités conditionnelles uniponctuelles, soit par des probabilités fini-conditionnelles uniponctuelles. Les résultats principaux sont les critères exprimés en termes de probabilités fini-conditionnelles uniponctuelles, l'un desquels peut être pris comme une définition purement probabiliste du champ aléatoire de Gibbs.

Key words: Gibbsianness, Gibbs random fields, Gibbsian specifications, one-point conditional distribution, one-point finite-conditional distribution.

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The classes of processes considered in the random processes theory are usually specified by some properties of their finite-dimensional or conditional distributions. However in practice, the study of a particular class usually goes through some representation theorem expressing processes in terms of simple and convenient objects, such as transition matrices for Markov chains, characteristic functions for processes with independent increments, spectral functions for stationary processes, and so on.

The situation is quite different for the class of Gibbs random fields. Historically, instead of being characterized by some properties of their finite-dimensional or conditional distributions, Gibbs random fields have been defined directly by representation of their conditional distributions in terms of potentials. And only afterwards the problem of probabilistic characterization of Gibbs random fields was considered.

It was shown in Kozlov [13] and Sullivan [17] that Gibbs random fields (with uniformly convergent potentials) can be characterized by strict positivity and quasilocality of their conditional distributions. More precisely, in order for a random field to be Gibbsian, its conditional distribution (which consists of conditional probabilities on finite volumes with conditions on the entire exterior and is defined up to a set of probability zero) must have a version which is a strictly positive quasilocal specification. As we see, this criterion imposes conditions on an object (conditional distribution) which is neither unambiguously defined, nor constructive (is infinite-dimensional), and, moreover, does not determine the random field uniquely (phase transitions). In our opinion, it is preferable that a characterization be in terms of an object which does not have these features.

As a matter of fact, such characterization already exists for the particular class of real-valued finite-range potentials. It was shown in Averintsev [1, 2, 3] and Sullivan [16] that Gibbs random fields (with such potentials) can be characterized by strict positivity and Markov properties. Note that for strictly positive random fields the Markov property can be formulated using only conditional probabilities on single sites with finite-volume conditions (see, for example, Suomela [18]). These probabilities are defined unambiguously and in constructive manner as ratios of finite-dimensional probabilities. Moreover, according to Dalalyan and Nahapetian [7], the system of all such probabilities, which we call one-point finite-conditional distribution, uniquely determines (can be identified with) the random field.

The aim of this paper is to characterize Gibbs random fields by some properties of their one-point finite-conditional distributions in the general case of uniformly convergent potentials. It is worth mentioning that such characterization is very natural in light of and was made possible due to the one-point framework developed in some recent papers. Namely, an approach towards description of random fields was developed by the authors in [4, 5, 6], where Dobrushin's well-known description of random fields by means of conditional distributions was reduced to description of random fields by means of one-point conditional distributions (the system of conditional probabilities on single sites with conditions on the entire exterior). Later on, a closely related and in some way complementary description of random fields based on one-point finite-conditional distributions was proposed in Dalalyan and Nahapetian [7].

The main outcome of the present work are random field Gibbsianness criteria in terms of one-point finite-conditional distribution. Let us emphasize that one of these criteria can be taken as a purely probabilistic definition of Gibbs random

field's conditional distribution. The criteria are established in Section 3, which also contains some additional results on characterization of Gibbsian specifications and Gibbs random fields, along with a brief survey of the known ones. Before that, some necessary notation and prerequisites are given in Section 1, while the above mentioned one-point framework is presented in Section 2.

1 Preliminaries

In this section we briefly recall some necessary notions and facts from the theory of Gibbs random fields.

1.1 Random fields

We consider random fields on the ν-dimensional integer lattice $\mathbb{Z}^\nu$, that is, probability measures $\mathbf{P}$ on $\left(\mathscr{X}^{\mathbb{Z}^\nu}, \mathscr{F}^{\mathbb{Z}^\nu}\right)$ where $(\mathscr{X}, \mathscr{F})$ is some measurable space of values on single sites (*state space*). Usually the space $\mathscr{X}$ is assumed to be endowed with some topology $\mathscr{T}$, and $\mathscr{F}$ is assumed to be the Borel σ-algebra for this topology. In this work we concentrate on the case when $\mathscr{X}$ is finite, $\mathscr{T}$ is the discrete topology, and $\mathscr{F}$ is the total σ-algebra, that is, $\mathscr{F} = \mathscr{T} = \operatorname{part}(\mathscr{X})$.

For any $S \subset \mathbb{Z}^\nu$, we denote $\mathscr{E}(S)$ the set of all finite subsets of S, that is, we put $\mathscr{E}(S) = \left\{\Lambda \subset S : |\Lambda| < \infty\right\}$ where $|\Lambda|$ is the number of points of the set Λ. For convenience of notation we will omit braces for one-point sets, that is, will write t instead of $\{t\}$. We put also $\mathscr{E}^*(S) = \mathscr{E} \setminus \{\emptyset\}$. For $S = \mathbb{Z}^\nu$ we write $\mathscr{E} = \mathscr{E}(\mathbb{Z}^\nu)$ and $\mathscr{E}^* = \mathscr{E}^*(\mathbb{Z}^\nu)$.

For any $S \subset \mathbb{Z}^\nu$, the space $\mathscr{X}^S$ is the space of all configurations on S. If $S = \emptyset$, we assume that the space $\mathscr{X}^\emptyset = \{\boldsymbol{\emptyset}\}$ where $\boldsymbol{\emptyset}$ is the empty configuration. For any $T, S \subset \mathbb{Z}^\nu$ such that $T \subset S$ and any configuration $\boldsymbol{x} = \{x_t,\ t \in S\}$ on S, we denote $\boldsymbol{x}_T$ the *subconfiguration* (*restriction*) of $\boldsymbol{x}$ on T defined by $\boldsymbol{x}_T = \{x_t,\ t \in T\}$. For any $T, S \subset \mathbb{Z}^\nu$ such that $T \cap S = \emptyset$ and any configurations $\boldsymbol{x}$ on T and $\boldsymbol{y}$ on S, we denote $\boldsymbol{xy}$ the *concatenation* of $\boldsymbol{x}$ and $\boldsymbol{y}$, that is, the configuration on $T \cup S$ equal to $\boldsymbol{x}$ on T and to $\boldsymbol{y}$ on S. For any configuration $\boldsymbol{x} \in \mathscr{X}^S$, the set $S \subset \mathbb{Z}^\nu$ will be called *support* of $\boldsymbol{x}$ and we will write $S = \mathfrak{S}(\boldsymbol{x})$. For any $\Lambda \in \mathscr{E}$, we denote

$$\widetilde{\mathscr{X}^\Lambda} = \bigcup_{\widetilde{\Lambda} \in \mathscr{E}^*(\Lambda^c)} \mathscr{X}^{\widetilde{\Lambda}}$$

the space of all configurations with non-empty finite support contained in the exterior of Λ.

For any $S \subset \mathbb{Z}^\nu$, a probability distribution on $\mathscr{X}^S$ will be denoted by $\mathbf{P}_S$. Note that if $S = \emptyset$ there exists only one probability distribution $\mathbf{P}_\emptyset(\boldsymbol{\emptyset}) = 1$. For any $T, S \subset \mathbb{Z}^\nu$ such that $T \subset S$ and any $\mathbf{P}_S$, we denote $\left(\mathbf{P}_S\right)_T$ the *marginal distribution* (*restriction*) of $\mathbf{P}_S$ on T. If $\Lambda \in \mathscr{E}$ and $I \subset \Lambda$, we can write $\mathbf{P}_\Lambda = \left\{\mathbf{P}_\Lambda(\boldsymbol{x}),\ \boldsymbol{x} \in \mathscr{X}^\Lambda\right\}$ and

$$\left(\mathbf{P}_\Lambda\right)_I(\boldsymbol{x}) = \sum_{\boldsymbol{y} \in \mathscr{X}^{\Lambda \setminus I}} \mathbf{P}_\Lambda(\boldsymbol{xy}), \quad \boldsymbol{x} \in \mathscr{X}^I.$$

Any random field $\mathbf{P}$ on $\mathbb{Z}^\nu$ is uniquely determined by (can be identified with) the system $\{\mathbf{P}_\Lambda,\ \Lambda \in \mathscr{E}\}$ of its *finite-dimensional distributions* which are consistent in the sense that for any $\Lambda \in \mathscr{E}$ and $I \subset \Lambda$ we have $\left(\mathbf{P}_\Lambda\right)_I = \mathbf{P}_I$.

dimensional distribution $\mathbf{P}_\Lambda$ is *strictly positive*, that is, $\mathbf{P}_\Lambda(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in \mathscr{X}^\Lambda$.

1.2 Finite-conditional and conditional distributions of random fields

Let $\mathbf{P}$ be some random field. For any $\Lambda \in \mathscr{E}$, we denote $\mathbf{P}_{\mathscr{E}^*(\Lambda^c)}$ the measure on $\widetilde{\mathscr{X}^\Lambda}$ whose projection on $\mathscr{X}^{\widetilde{\Lambda}}$ is $\mathbf{P}_{\widetilde{\Lambda}}$ for any $\widetilde{\Lambda} \in \mathscr{E}^*(\Lambda^c)$, that is, $\mathbf{P}_{\mathscr{E}^*(\Lambda^c)}$ is the direct sum of the measures $\mathbf{P}_{\widetilde{\Lambda}}$.

For all $\Lambda \in \mathscr{E}$, the ratios

$$\boldsymbol{q}_\Lambda^{\widetilde{\boldsymbol{x}}}(\boldsymbol{x}) = \frac{\mathbf{P}_{\Lambda\cup\mathfrak{S}(\widetilde{\boldsymbol{x}})}(\boldsymbol{x}\widetilde{\boldsymbol{x}})}{\mathbf{P}_{\mathfrak{S}(\widetilde{\boldsymbol{x}})}(\widetilde{\boldsymbol{x}})}\ , \quad \boldsymbol{x} \in \mathscr{X}^\Lambda,$$

exist for $\mathbf{P}_{\mathscr{E}^*(\Lambda^c)}$-almost all $\widetilde{\boldsymbol{x}} \in \widetilde{\mathscr{X}^\Lambda}$. Any system

$$\widetilde{\boldsymbol{\mathcal{Q}}} = \left\{ \mathbf{Q}_\Lambda^{\widetilde{\boldsymbol{x}}}, \quad \Lambda \in \mathscr{E} \text{ and } \widetilde{\boldsymbol{x}} \in \widetilde{\mathscr{X}^\Lambda} \right\}$$

of probability distributions such that for every $\Lambda \in \mathscr{E}$ we have $\mathbf{Q}_\Lambda^{\widetilde{\boldsymbol{x}}} = \boldsymbol{q}_\Lambda^{\widetilde{\boldsymbol{x}}}$ for $\mathbf{P}_{\mathscr{E}^*(\Lambda^c)}$-almost all $\widetilde{\boldsymbol{x}} \in \widetilde{\mathscr{X}^\Lambda}$ will be called *finite-conditional distribution* of the random field $\mathbf{P}$. The subsystem of $\widetilde{\boldsymbol{\mathcal{Q}}}$ consisting of single-site distributions $\big(|\Lambda| = 1\big)$ will be called *one-point finite-conditional distribution* of $\mathbf{P}$. Note that in general a random field may have many *versions* both of finite-conditional and one-point finite-conditional distributions. However, for strictly positive random fields these distributions are uniquely determined and consist of strictly positive elements. If this positivity is uniform with respect to $\widetilde{\boldsymbol{x}}$, the (one-point) finite-conditional distribution will be called *uniformly nonnull*.

Further, for all $\Lambda \in \mathscr{E}$, the limits

$$\boldsymbol{q}_\Lambda^{\overline{\boldsymbol{x}}}(\boldsymbol{x}) = \lim_{\widetilde{\Lambda}\uparrow\mathbb{Z}^\nu\setminus\Lambda} \boldsymbol{q}_\Lambda^{\overline{\boldsymbol{x}}_{\widetilde{\Lambda}}}(\boldsymbol{x})\ , \quad \boldsymbol{x} \in \mathscr{X}^\Lambda,$$

exist for $\mathbf{P}_{\Lambda^c}$-almost all $\overline{\boldsymbol{x}} \in \mathscr{X}^{\Lambda^c}$. Any system

$$\boldsymbol{\mathcal{Q}} = \left\{ \mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}}, \quad \Lambda \in \mathscr{E} \text{ and } \overline{\boldsymbol{x}} \in \mathscr{X}^{\Lambda^c} \right\}$$

of probability distributions such that for every $\Lambda \in \mathscr{E}$ we have $\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}} = \boldsymbol{q}_\Lambda^{\overline{\boldsymbol{x}}}$ for $\mathbf{P}_{\Lambda^c}$-almost all $\overline{\boldsymbol{x}} \in \mathscr{X}^{\Lambda^c}$ will be called *conditional distribution* of the random field $\mathbf{P}$. The subsystem of $\boldsymbol{\mathcal{Q}}$ consisting of single-site distributions will be called *one-point conditional distribution* of $\mathbf{P}$. Note that in general a random field $\mathbf{P}$ may have many *versions* both of conditional and one-point conditional distributions (even if $\mathbf{P}$ is strictly positive). Note also that if a random field $\mathbf{P}$ has a strictly positive version of conditional distribution (a version all of whose elements are strictly positive), then $\mathbf{P}$ is necessarily strictly positive itself.

Concluding this section let us remark that (one-point) conditional distribution can be deduced from (one-point) finite-conditional distribution, but not the other way around. So, the latter contains "more complete" information about the random field than the former. Especially it becomes apparent in the Markov case, when (one-point) conditional distribution can be considered as a subsystem of (one-point)

denote the neighborhood of the set Λ. As we have $\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}} = \mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}_{\partial\Lambda}}$, the elements of (one-point) conditional distribution of $\mathbf{P}$ can also be considered as elements of (one-point) finite-conditional distribution of $\mathbf{P}$. The converse is not true since not all the elements of the latter correspond to the elements of the former, but only the elements $\mathbf{Q}_\Lambda^{\widetilde{\boldsymbol{x}}}$ such that $\mathfrak{S}(\widetilde{\boldsymbol{x}}) \supset \partial\Lambda$.

1.3 Description of random fields by means of conditional distributions

The well-known description of random fields by means of conditional distributions introduced by Dobrushin in [8, 9, 10] is carried out in terms of specifications. A system

$$\boldsymbol{\mathcal{Q}} = \left\{ \mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}}, \quad \Lambda \in \mathscr{E} \text{ and } \overline{\boldsymbol{x}} \in \mathscr{X}^{\Lambda^c} \right\}$$

of probability distributions is called *specification* if

$$\begin{gathered} \mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}}(\boldsymbol{x}\boldsymbol{y}) = \left(\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}}\right)_{\Lambda \setminus I}(\boldsymbol{x}) \; \mathbf{Q}_I^{\overline{\boldsymbol{x}}\boldsymbol{x}}(\boldsymbol{y}) \\ \text{for all } \; \Lambda \in \mathscr{E}, \;\; I \subset \Lambda, \;\; \boldsymbol{x} \in \mathscr{X}^{\Lambda \setminus I}, \;\; \boldsymbol{y} \in \mathscr{X}^I \;\text{ and }\; \overline{\boldsymbol{x}} \in \mathscr{X}^{\Lambda^c}. \end{gathered} \tag{1}$$

Note that any version of conditional distribution of a random field $\mathbf{P}$ satisfies a somewhat weaker than (1) condition, where $\mathbf{P}_{\Lambda^c}$-almost all (and not necessarily all) $\overline{\boldsymbol{x}} \in \mathscr{X}^{\Lambda^c}$ are considered. However, any random field possesses at least one version of conditional distribution being a specification (see Goldstein [12], Preston [14] and Sokal [15]).

One of the main goals of Dobrushin's theory is to study the set of all random fields *compatible* with a given specification, that is, having it as a version of conditional distribution. The best-known sufficient conditions for existence and for uniqueness of random fields compatible with a given specification are quasilocality and Dobrushin's uniqueness conditions respectively. The first one will play an important role in our considerations, so we recall it below.

Let $S \subset \mathbb{Z}^\nu$. A real-valued function g on $\mathscr{X}^S$ is called *quasilocal* if

$$\lim_{\Lambda \uparrow S} \; \sup_{\boldsymbol{x},\boldsymbol{y} \in \mathscr{X}^S \,:\, \boldsymbol{x}_\Lambda = \boldsymbol{y}_\Lambda} \left| g(\boldsymbol{x}) - g(\boldsymbol{y}) \right| = 0,$$

or equivalently if g is a uniform limit of functions depending only on values of configuration on finite sets of sites (*local* functions). Note also that the quasilocality is nothing but continuity with respect to the topology $\mathscr{T}^S$ and, taking into account that $\mathscr{X}^S$ is compact, the strict positivity and uniform nonnullness conditions are equivalent for quasilocal functions.

A specification $\boldsymbol{\mathcal{Q}} = \left\{ \mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}}, \quad \Lambda \in \mathscr{E} \text{ and } \overline{\boldsymbol{x}} \in \mathscr{X}^{\Lambda^c} \right\}$ is called *(quasi)local* if for any $\Lambda \in \mathscr{E}$ and $\boldsymbol{x} \in \mathscr{X}^\Lambda$ the function $\overline{\boldsymbol{x}} \mapsto \mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}}(\boldsymbol{x})$ on $\mathscr{X}^{\Lambda^c}$ is (quasi)local.

Finally, a specification will be called *strictly positive* if all its elements are strictly positive.

1.4 Gibbs random fields and Gibbsian specifications

The main object of consideration of the present paper are Gibbs random fields. The latters are defined in terms of Gibbsian specifications, which in turn are defined in terms of potentials.

Any function Φ on $\mathscr{S}$ taking values in $\mathbb{R} \cup \{+\infty\}$ is called *(interaction) potential*. A potential Φ is called *convergent* if it is real-valued and the series

$$\sum_{\widetilde{J} \in \mathscr{E}(t^{\mathrm{c}})} \Phi\big(x \overline{\boldsymbol{x}}_{\widetilde{J}}\big) \tag{2}$$

converge for all $t \in \mathbb{Z}^\nu$, $x \in \mathscr{X}^t$ and $\overline{\boldsymbol{x}} \in \mathscr{X}^{t^{\mathrm{c}}}$.

A potential Φ is called *uniformly convergent* if it is convergent and the convergence in (2) is uniform with respect to $\overline{\boldsymbol{x}}$.

A potential Φ is called *finite-range potential* if for any $t \in \mathbb{Z}^\nu$ there exist only a finite number of sets $\widetilde{J} \in \mathscr{E}(t^{\mathrm{c}})$ such that $\Phi \not\equiv 0$ on $\mathscr{X}^{t \cup \widetilde{J}}$. Note that any real-valued finite-range potential is uniformly convergent.

For an arbitrary convergent potential Φ one can construct the specification $\boldsymbol{\mathcal{Q}} = \big\{\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}}, \quad \Lambda \in \mathscr{E} \text{ and } \overline{\boldsymbol{x}} \in \mathscr{X}^{\Lambda^{\mathrm{c}}}\big\}$ given by Gibbs formulae

$$\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}}(\boldsymbol{x}) = \frac{\exp\big(-U_\Lambda^{\overline{\boldsymbol{x}}}(\boldsymbol{x})\big)}{\sum\limits_{\boldsymbol{y} \in \mathscr{X}^\Lambda} \exp\big(-U_\Lambda^{\overline{\boldsymbol{x}}}(\boldsymbol{y})\big)}\,, \quad \Lambda \in \mathscr{E},\ \boldsymbol{x} \in \mathscr{X}^\Lambda,\ \overline{\boldsymbol{x}} \in \mathscr{X}^{\Lambda^{\mathrm{c}}}, \tag{3}$$

where

$$U_\Lambda^{\overline{\boldsymbol{x}}}(\boldsymbol{x}) = \sum_{J\,:\,\emptyset \neq J \subset \Lambda} \ \sum_{\widetilde{J} \in \mathscr{E}(\Lambda^{\mathrm{c}})} \Phi\big(\boldsymbol{x}_J \overline{\boldsymbol{x}}_{\widetilde{J}}\big), \quad \Lambda \in \mathscr{E},\ \boldsymbol{x} \in \mathscr{X}^\Lambda,\ \overline{\boldsymbol{x}} \in \mathscr{X}^{\Lambda^{\mathrm{c}}}. \tag{4}$$

The specification $\boldsymbol{\mathcal{Q}}$ is called *Gibbsian with potential* Φ. Any random field compatible with $\boldsymbol{\mathcal{Q}}$ is called *Gibbs random field with potential* Φ.

In this paper we consider uniformly convergent potentials only, so Gibbsian specifications and Gibbs random fields with uniformly convergent potentials will be called shortly *Gibbsian specifications* and *Gibbs random fields* correspondingly.

2 One-point framework

The idea that it is possible to describe and study random fields using only one-point conditional probabilities goes back to Dobrushin [8]. It was realized in the authors' works [4, 5, 6] using one-point conditional distributions, and in Dalalyan and Nahapetian [7] using one-point finite-conditional distributions.

2.1 Description of specifications and random fields by means of one-point conditional distributions

In this section we briefly recall the main results of the authors' works [4, 5, 6].

In these papers, under wide positivity assumptions (*very weak positivity*) a necessary and sufficient conditions for a system $\big\{\mathbf{Q}_t^{\overline{\boldsymbol{x}}}, \quad t \in \mathbb{Z}^\nu \text{ and } \overline{\boldsymbol{x}} \in \mathscr{X}^{t^{\mathrm{c}}}\big\}$ of probability distributions to be contained in some specification were established. A system satisfying these conditions was called 1-specification. It was equally shown that the specification containing the given 1-specification is uniquely determined by some explicit formulae involving only the elements of this 1-specification. Moreover, since these formulae make use of finite number of elementary operations, the entire specification is quasilocal if and only if the 1-specification is, and the set of random fields

ble with the entire specification. So, whole Dobrushin's theory can be reformulated in terms of 1-specifications and one can speak about description of random fields by means of one-point conditional distributions.

Let us now give some more details in the particular strictly positive case.

The definition of strictly positive 1-specification can be formulated in the following way: a system

$$\boldsymbol{\mathcal{Q}} = \left\{ \mathbf{Q}_t^{\overline{\boldsymbol{x}}}, \quad t \in \mathbb{Z}^\nu \text{ and } \overline{\boldsymbol{x}} \in \mathscr{X}^{t^c} \right\}$$

of strictly positive probability distributions will be called *1-specification* if

$$\begin{gathered} \mathbf{Q}_t^{\overline{\boldsymbol{x}}v}(x)\ \mathbf{Q}_s^{\overline{\boldsymbol{x}}x}(y)\ \mathbf{Q}_t^{\overline{\boldsymbol{x}}y}(u)\ \mathbf{Q}_s^{\overline{\boldsymbol{x}}u}(v) = \mathbf{Q}_s^{\overline{\boldsymbol{x}}u}(y)\ \mathbf{Q}_t^{\overline{\boldsymbol{x}}y}(x)\ \mathbf{Q}_s^{\overline{\boldsymbol{x}}x}(v)\ \mathbf{Q}_t^{\overline{\boldsymbol{x}}v}(u) \\ \text{for all } \ t, s \in \mathbb{Z}^\nu,\ \ x, u \in \mathscr{X}^t,\ \ y, v \in \mathscr{X}^s \ \text{ and } \ \overline{\boldsymbol{x}} \in \mathscr{X}^{\{t,s\}^c}. \end{gathered} \tag{5}$$

Further, a 1-specification $\boldsymbol{\mathcal{Q}} = \left\{ \mathbf{Q}_t^{\overline{\boldsymbol{x}}}, \quad t \in \mathbb{Z}^\nu \text{ and } \overline{\boldsymbol{x}} \in \mathscr{X}^{t^c} \right\}$ is called *(quasi)local* if for any $t \in \mathbb{Z}^\nu$ and $x \in \mathscr{X}^t$ the function $\overline{\boldsymbol{x}} \mapsto \mathbf{Q}_t^{\overline{\boldsymbol{x}}}(x)$ on $\mathscr{X}^{t^c}$ is (quasi)local. Finally, a random fields $\mathbf{P}$ is called *compatible* with a 1-specification if the latter is a version of one-point conditional distribution of $\mathbf{P}$.

The above mentioned explicit formulae determining the elements of the specification $\boldsymbol{\mathcal{Q}} = \left\{ \mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}}, \quad \Lambda \in \mathscr{E} \text{ and } \overline{\boldsymbol{x}} \in \mathscr{X}^{\Lambda^c} \right\}$ containing the given strictly positive 1-specification have the following form: for all $\Lambda \in \mathscr{E}$ and $\overline{\boldsymbol{x}} \in \mathscr{X}^\Lambda$ one has

$$\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}}(\boldsymbol{x}) = \frac{\mathbf{Q}_{t_1}^{\overline{\boldsymbol{x}}\boldsymbol{u}_{\{t_2,\dots,t_n\}}}(x_{t_1})\ \mathbf{Q}_{t_2}^{\overline{\boldsymbol{x}}\boldsymbol{x}_{\{t_1\}}\boldsymbol{u}_{\{t_3,\dots,t_n\}}}(x_{t_2})\ \cdots\ \mathbf{Q}_{t_n}^{\overline{\boldsymbol{x}}\boldsymbol{x}_{\{t_1,\dots,t_{n-1}\}}}(x_{t_n})}{\mathbf{Q}_{t_1}^{\overline{\boldsymbol{x}}\boldsymbol{u}_{\{t_2,\dots,t_n\}}}(u_{t_1})\ \mathbf{Q}_{t_2}^{\overline{\boldsymbol{x}}\boldsymbol{x}_{\{t_1\}}\boldsymbol{u}_{\{t_3,\dots,t_n\}}}(u_{t_2})\ \cdots\ \mathbf{Q}_{t_n}^{\overline{\boldsymbol{x}}\boldsymbol{x}_{\{t_1,\dots,t_{n-1}\}}}(u_{t_n})} \times C, \quad \boldsymbol{x} \in \mathscr{X}^\Lambda,$$

where C is the normalizing factor. Here some fixed configuration $\boldsymbol{u} \in \mathscr{X}^\Lambda$ and some enumeration $t_1, \dots, t_n$ of elements of Λ are chosen arbitrary. Note that the right hand side of these formulae does not depend on this choice (correctness of the formulae) thanks to consistency condition (5). Note also, that these formulae imply that the specification containing a strictly positive 1-specification is necessarily strictly positive itself.

2.2 Description of random fields by means of one-point finite-conditional distributions

Now we turn to the problem of description of random fields by means of one-point finite-conditional distributions considered in Dalalyan and Nahapetian [7]. This description is closely related (and in some way complementary) to the one presented in the previous section.

First, let us note that the necessary and sufficient conditions for a system $\widetilde{\boldsymbol{q}} = \left\{ \mathbf{Q}_t^{\widetilde{\boldsymbol{x}}}, \quad t \in \mathbb{Z}^\nu \text{ and } \widetilde{\boldsymbol{x}} \in \widetilde{\mathscr{X}^t} \right\}$ of probability distributions to be contained in some system $\widetilde{\boldsymbol{\mathcal{Q}}} = \left\{ \mathbf{Q}_\Lambda^{\widetilde{\boldsymbol{x}}}, \quad \Lambda \in \mathscr{E} \text{ and } \widetilde{\boldsymbol{x}} \in \widetilde{\mathscr{X}^\Lambda} \right\}$ of probability distributions satisfying

$$\begin{gathered} \mathbf{Q}_\Lambda^{\widetilde{\boldsymbol{x}}}(\boldsymbol{x}\boldsymbol{y}) = \mathbf{Q}_{\Lambda\setminus I}^{\widetilde{\boldsymbol{x}}}(\boldsymbol{x})\ \mathbf{Q}_I^{\widetilde{\boldsymbol{x}}\boldsymbol{x}}(\boldsymbol{y}) \\ \text{for all } \ \Lambda \in \mathscr{E},\ \ I \subset \Lambda,\ \ \boldsymbol{x} \in \mathscr{X}^{\Lambda\setminus I},\ \ \boldsymbol{y} \in \mathscr{X}^I \ \text{ and } \ \widetilde{\boldsymbol{x}} \in \widetilde{\mathscr{X}^\Lambda} \end{gathered} \tag{6}$$

are the following:

$$\begin{gathered} \mathbf{Q}_t^{\widetilde{\boldsymbol{x}}}(x)\ \mathbf{Q}_s^{\widetilde{\boldsymbol{x}}x}(y) = \mathbf{Q}_s^{\widetilde{\boldsymbol{x}}}(y)\ \mathbf{Q}_t^{\widetilde{\boldsymbol{x}}y}(x) \\ \text{for all } \ t, s \in \mathbb{Z}^\nu,\ \ x \in \mathscr{X}^t,\ \ y \in \mathscr{X}^s \ \text{ and } \ \widetilde{\boldsymbol{x}} \in \widetilde{\mathscr{X}^{\{t,s\}}}. \end{gathered} \tag{7}$$

distribution of some strictly positive random field, then it necessarily satisfies the condition (7) $\big($the condition (6)$\big)$. However, in order for a strictly positive system $\widetilde{\boldsymbol{q}}$ satisfying (7) $\big(\widetilde{\boldsymbol{\mathcal{Q}}}$ satisfying (6)$\big)$ to be the one-point finite-conditional $\big($the finite-conditional$\big)$ distribution of some strictly positive random field one needs some additional conditions. It turns out that such conditions are the following:

$$\begin{gathered}\mathbf{Q}_t^v(x)\ \mathbf{Q}_s^x(y)\ \mathbf{Q}_t^y(u)\ \mathbf{Q}_s^u(v) = \mathbf{Q}_s^u(y)\ \mathbf{Q}_t^y(x)\ \mathbf{Q}_s^x(v)\ \mathbf{Q}_t^v(u) \\ \text{for all}\ \ t, s \in \mathbb{Z}^\nu,\ \ x, u \in \mathscr{X}^t \ \text{ and }\ y, v \in \mathscr{X}^s.\end{gathered} \tag{8}$$

More precisely, in [7] it was shown that the strict positivity of elements and the fulfillment of the conditions (7) and (8) are necessary and sufficient for a system $\big\{\mathbf{Q}_t^{\widetilde{\boldsymbol{x}}},\quad t \in \mathbb{Z}^\nu$ and $\widetilde{\boldsymbol{x}} \in \widetilde{\mathscr{X}^t}\big\}$ of probability distributions to be the one-point finite-conditional distribution of some strictly positive random field. It was equally shown that this random field is uniquely determined by this system. In particular, a strictly positive random field is uniquely determined by (can be identified with) its one-point finite-conditional distribution, and so one can speak about description of random fields by means of one-point finite-conditional distributions.

3 Gibbsianness criteria

In this section we turn to the main subject of the present work: the problem of probabilistic characterization of Gibbs random fields. But first, let us mention several results concerning Gibbsianness of specifications. These results are useful since taking into account the definition of Gibbs random field, one can transform them into random field Gibbsianness criteria in terms of conditional distribution.

3.1 Specification Gibbsianness criteria

The problem of characterization of the class of Gibbsian specifications with potentials satisfying some given conditions was subject of consideration of many authors: one can refer to Averintsev [1, 2, 3] and Sullivan [16] for real-valued finite-range potentials, Kozlov [13] and Sullivan [17] for uniformly convergent potentials, our works [4, 5] for more general potentials (which in particular can assume the value $+\infty$).

Concerning Gibbsian specifications (with uniformly convergent potentials), the best-known criterion is the following one $\big($see, for example, Georgii [11]$\big)$.

Criterion 1 [Kozlov-Sullivan]. — *A specification is Gibbsian if and only if it is quasilocal and strictly positive.*

Combining this criterion with the results of Section 2.1, one clearly gets the following characterization already obtained by the authors in [4, 5].

Criterion 2. — *A specification is Gibbsian if and only if the 1-specification contained in it is quasilocal and strictly positive.*

Since the uniform convergence of potential assures the quasilocality of the 1-specification expressed by Gibbs formulae, one can also obtain the following corollary of Criterion 2.

tained in it can be expressed by Gibbs formulae (3) *and* (4) *with some uniformly convergent potential.*

Concerning Gibbsian specifications with real-valued finite-range potentials, let us recall that they are characterized by strict positivity and locality of their single-site parts, and so Gibbs random fields with such potentials are characterized by strict positivity and Markov properties. Since for strictly positive random fields the Markov property can be formulated using only conditional probabilities on single sites with finite-volume conditions $\big($see, for example, Suomela [18]$\big)$, one clearly has a characterization of Gibbs random fields with real-valued finite-range potentials in terms of one-point finite-conditional distribution.

Establishment of a similar characterization in the general case of uniformly convergent potentials is not so straightforward and will be accomplished in Sections 3.4 and 3.5. Before that, random field Gibbsianness criteria in terms of conditional and one-point conditional distribution are obtained in the next two sections by means of transformation and subsequent improvement of Criteria 1–3.

3.2 Random field Gibbsianness criteria in terms of conditional distribution

Combining the definition of Gibbs random field with Criterion 1 one gets the following well-known characterization: a random field is a Gibbs random field if and only if it has a version of conditional distribution which is a strictly positive quasilocal specification. This criterion can be improved in the following way.

Criterion 4. — *A random field is a Gibbs random field if and only if it has a version of conditional distribution which is quasilocal and strictly positive.*

Since the strict positivity of a version of conditional distribution implies the strict positivity of the random field, the criterion is immediately deduced from the following proposition which is of general interest.

Proposition 1. — *If a strictly positive random field has a quasilocal version of conditional distribution, the latter is unique and is necessarily a specification.*

Proof. Let $\mathbf{P}$ be a strictly positive random field. First, note that the measure $\mathbf{P}$ is everywhere dense, that is, $\mathbf{P}(A) > 0$ for any non-empty open set $A \in \mathscr{T}^{\mathbb{Z}^\nu} \setminus \{\varnothing\}$. Indeed, since such a set A necessarily contains a non-empty cylinder subset A', which in turn contains a subset $\big\{\overline{\boldsymbol{x}} \in \mathscr{X}^{\mathbb{Z}^\nu} : \overline{\boldsymbol{x}}_\Lambda = \boldsymbol{x}^\circ\big\}$ where $\Lambda \in \mathscr{E}^*$ and $\boldsymbol{x}^\circ \in \mathscr{X}^\Lambda$, we have $\mathbf{P}(A) \geqslant \mathbf{P}(A') \geqslant \mathbf{P}_\Lambda(\boldsymbol{x}^\circ) > 0$. An important evident property of everywhere dense measures is the following: if a continuous function is equal to zero almost everywhere (with respect to such a measure), then it is equal to zero everywhere.

Now, suppose $\big\{\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}},\quad \Lambda \in \mathscr{E} \text{ and } \overline{\boldsymbol{x}} \in \mathscr{X}^{\Lambda^c}\big\}$ and $\big\{\boldsymbol{q}_\Lambda^{\overline{\boldsymbol{x}}},\quad \Lambda \in \mathscr{E} \text{ and } \overline{\boldsymbol{x}} \in \mathscr{X}^{\Lambda^c}\big\}$ are two quasilocal versions of conditional distribution of $\mathbf{P}$. Hence, for any $\Lambda \in \mathscr{E}$ and $\boldsymbol{x} \in \mathscr{X}^\Lambda$, the function $\overline{\boldsymbol{x}} \mapsto \mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}}(\boldsymbol{x}) - \boldsymbol{q}_\Lambda^{\overline{\boldsymbol{x}}}(\boldsymbol{x})$ on $\mathscr{X}^{\Lambda^c}$ is quasilocal and equal to zero $\mathbf{P}_{\Lambda^c}$-almost everywhere. Since quasilocality is nothing but continuity and the measure $\mathbf{P}_{\Lambda^c}$ is everywhere dense, this function is equal to zero everywhere. So, the uniqueness is proved.

Finally, suppose $\boldsymbol{\mathcal{Q}} = \big\{\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}},\quad \Lambda \in \mathscr{E} \text{ and } \overline{\boldsymbol{x}} \in \mathscr{X}^{\Lambda^c}\big\}$ is (the unique) quasilocal version of conditional distribution of $\mathbf{P}$. For any $\Lambda \in \mathscr{E}$, $I \subset \Lambda$, $\boldsymbol{x} \in \mathscr{X}^{\Lambda \setminus I}$ and $\boldsymbol{y} \in \mathscr{X}^I$ consider the function $\overline{\boldsymbol{x}} \mapsto \mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}}(\boldsymbol{x}\boldsymbol{y}) - \big(\mathbf{Q}_\Lambda^{\overline{\boldsymbol{x}}}\big)_{\Lambda \setminus I}(\boldsymbol{x})\, \mathbf{Q}_I^{\overline{\boldsymbol{x}}\boldsymbol{x}}(\boldsymbol{y})$ on $\mathscr{X}^{\Lambda^c}$. This

probabilities, is equal to zero $\mathbf{P}_{\Lambda^c}$-almost everywhere. Hence it is equal to zero everywhere, and so $\boldsymbol{Q}$ is a specification. □

Let us note that Criterion 4 was as a matter of fact obtained in Sullivan [17] using a different approach.

3.3 Random field Gibbsianness criteria in terms of one-point conditional distribution

Criterion 4 characterizes Gibbs random fields in terms of conditional distribution. However, in view of Section 2.1, it should be possible to do it in terms of one-point conditional distribution.

Indeed, combining the definition of Gibbs random field with Criterion 2 and taking into account the results of Section 2.1, one gets the following characterization: a random field is a Gibbs random field if and only if it has a version of one-point conditional distribution which is a strictly positive quasilocal 1-specification. As in the preceding section we can improve this criterion in the following way.

Criterion 5. — *A random field is a Gibbs random field if and only if it has a version of one-point conditional distribution which is quasilocal and strictly positive.*

The criterion is immediately deduced from the following two propositions which are of general interest.

Proposition 2. — *If a random field* $\mathbf{P}$ *has a strictly positive version of one-point conditional distribution, then* $\mathbf{P}$ *is strictly positive itself.*

Proof. Let us suppose that the random field $\mathbf{P}$ is not strictly positive. In this case we can find some $\Lambda \in \mathscr{E}^*$, $t \in \Lambda$ and $\boldsymbol{z} \in \mathscr{X}^{\Lambda}$ such that $\mathbf{P}_{\Lambda}(\boldsymbol{z}) = 0$ and $\mathbf{P}_{\Lambda\setminus t}(\boldsymbol{z}_{\Lambda\setminus t}) > 0$ $\big($recall that $\mathbf{P}_{\emptyset}(\boldsymbol{\phi}) = 1\big)$. Now denote

$$A = \left\{ \overline{\boldsymbol{x}} \in \mathscr{X}^{t^c} : \overline{\boldsymbol{x}}_{\Lambda\setminus t} = \boldsymbol{z}_{\Lambda\setminus t} \right\}.$$

Obviously $\mathbf{P}_{t^c}(A) = \mathbf{P}_{\Lambda\setminus t}(\boldsymbol{z}_{\Lambda\setminus t}) > 0$. Introduce also

$$B = \bigcap_{\widetilde{\Lambda} \in \mathscr{E}(t^c)} \left\{ \overline{\boldsymbol{x}} \in \mathscr{X}^{t^c} : \mathbf{P}_{\widetilde{\Lambda}}(\overline{\boldsymbol{x}}_{\widetilde{\Lambda}}) > 0 \right\}.$$

Since B is a countable intersection of sets of probability 1, we have $\mathbf{P}_{t^c}(B) = 1$. So, it comes $\mathbf{P}_{t^c}(A \cap B) > 0$.

For all $\overline{\boldsymbol{x}} \in A \cap B$ and all $\widetilde{\Lambda} \in \mathscr{E}(t^c)$ such that $\widetilde{\Lambda} \supset \Lambda \setminus t$, we have

$$\boldsymbol{q}_t^{\overline{\boldsymbol{x}}_{\widetilde{\Lambda}}}(\boldsymbol{z}_t) \triangleq \frac{\mathbf{P}_{t\cup\widetilde{\Lambda}}(\boldsymbol{z}_t \overline{\boldsymbol{x}}_{\widetilde{\Lambda}})}{\mathbf{P}_{\widetilde{\Lambda}}(\overline{\boldsymbol{x}}_{\widetilde{\Lambda}})} = 0.$$

Hence, for all $\overline{\boldsymbol{x}} \in A \cap B$ we get

$$\lim_{\widetilde{\Lambda} \uparrow \mathbb{Z}^\nu \setminus t} \boldsymbol{q}_t^{\overline{\boldsymbol{x}}_{\widetilde{\Lambda}}}(\boldsymbol{z}_t) = 0$$

which contradicts the existence of a strictly positive version of one-point conditional distribution of $\mathbf{P}$. □

of one-point conditional distribution, the latter is unique and is necessarily a 1-specification.

Proof. The uniqueness is proved following exactly the same argument as in the proof of Proposition 1.

To prove the second assertion, suppose $\left\{\mathbf{Q}_t^{\overline{\boldsymbol{x}}},\quad t\in\mathbb{Z}^\nu \text{ and } \overline{\boldsymbol{x}}\in\mathscr{X}^{t^{\mathrm{c}}}\right\}$ is (the unique) quasilocal version of one-point conditional distribution of a strictly positive random field $\mathbf{P}$. For any $t,s\in\mathbb{Z}^\nu$, $x,u\in\mathscr{X}^t$ and $y,v\in\mathscr{X}^s$ consider the function $\overline{\boldsymbol{x}}\mapsto \mathbf{Q}_t^{\overline{\boldsymbol{x}}v}(x)\,\mathbf{Q}_s^{\overline{\boldsymbol{x}}x}(y)\,\mathbf{Q}_t^{\overline{\boldsymbol{x}}y}(u)\,\mathbf{Q}_s^{\overline{\boldsymbol{x}}u}(v)-\mathbf{Q}_s^{\overline{\boldsymbol{x}}u}(y)\,\mathbf{Q}_t^{\overline{\boldsymbol{x}}y}(x)\,\mathbf{Q}_s^{\overline{\boldsymbol{x}}x}(v)\,\mathbf{Q}_t^{\overline{\boldsymbol{x}}v}(u)$ on $\mathscr{X}^{\{t,s\}^{\mathrm{c}}}$. Applying the reasoning used in the proof of Proposition 1, it clearly comes that this function is equal to zero everywhere. □

Concluding this section, let us note that combining the definition of Gibbs random field with Criterion 3 and taking into account the results of Section 2.1, one gets the following characterization.

Criterion 6. — *A random field is a Gibbs random field if and only if it has a version of one-point conditional distribution which can be expressed by Gibbs formulae* (3) *and* (4) *with some uniformly convergent potential.*

3.4 Random field Gibbsianness criteria in terms of one-point finite-conditional distribution I

Now we can establish random field Gibbsianness criteria in terms of one-point finite-conditional distribution, which are precisely the main outcome of the present paper. The first such criterion is the following.

Criterion 7. — *A random field is a Gibbs random field if and only if it is strictly positive and its one-point finite-conditional distribution* $\left\{\boldsymbol{q}_t^{\widetilde{\boldsymbol{x}}},\quad t\in\mathbb{Z}^\nu \text{ and } \widetilde{\boldsymbol{x}}\in\widetilde{\mathscr{X}^t}\right\}$ *satisfy one of the following equivalent conditions:*

(A) *the limits*
$$\lim_{\Lambda\uparrow\mathbb{Z}^\nu\setminus t}\boldsymbol{q}_t^{\overline{\boldsymbol{x}}_\Lambda}(x)\ ,\quad t\in\mathbb{Z}^\nu,\ x\in\mathscr{X}^t,\ \overline{\boldsymbol{x}}\in\mathscr{X}^{t^{\mathrm{c}}},$$
exist, are nonnull uniformly with respect to $\overline{\boldsymbol{x}}$*, and the convergence is uniform with respect to* $\overline{\boldsymbol{x}}$*,*

(B) *the limits*
$$\lim_{\Lambda\uparrow\mathbb{Z}^\nu\setminus t}\boldsymbol{q}_t^{\overline{\boldsymbol{x}}_\Lambda}(x)\ ,\quad t\in\mathbb{Z}^\nu,\ x\in\mathscr{X}^t,\ \overline{\boldsymbol{x}}\in\mathscr{X}^{t^{\mathrm{c}}},$$
exist, are strictly positive, and the convergence is uniform with respect to $\overline{\boldsymbol{x}}$*.*

Proof. The sufficiency is quite evident. Indeed, the strictly positive limits supposed to exist form a strictly positive version of one-point conditional distribution of the random field. The uniformity of convergence guarantees that this version is quasilocal and so, the sufficiency follows from Criterion 5. Let us also note that at the same time this quasilocality clearly yields the equivalence of the conditions (A) and (B).

Now let us turn to the proof of the necessity. Let $\mathbf{P}$ be a Gibbs random field. According to Criterion 5 it has a quasilocal and strictly positive version

conclude the proof it is sufficient to show that

$$\lim_{\Lambda\uparrow\mathbb{Z}^\nu\setminus t}\ \sup_{\overline{\boldsymbol{x}}\in\mathscr{X}^{t^{\mathrm{c}}}} \left|\boldsymbol{q}_t^{\overline{\boldsymbol{x}}_\Lambda}(x)-\mathbf{Q}_t^{\overline{\boldsymbol{x}}}(x)\right|=0$$

for all $t\in\mathbb{Z}^\nu$ and $x\in\mathscr{X}^t$.

For this we need the following inequality due to Sullivan:

$$\inf_{\overline{\boldsymbol{y}}\in\mathscr{X}^{t^{\mathrm{c}}}:\overline{\boldsymbol{y}}_\Lambda=\boldsymbol{z}} \mathbf{Q}_t^{\overline{\boldsymbol{y}}}(x)\leqslant \boldsymbol{q}_t^{\boldsymbol{z}}(x)\leqslant \sup_{\overline{\boldsymbol{y}}\in\mathscr{X}^{t^{\mathrm{c}}}:\overline{\boldsymbol{y}}_\Lambda=\boldsymbol{z}} \mathbf{Q}_t^{\overline{\boldsymbol{y}}}(x) \tag{9}$$

for all $t\in\mathbb{Z}^\nu$, $\Lambda\in\mathscr{E}^*(t^{\mathrm{c}})$, $x\in\mathscr{X}^t$ and $\boldsymbol{z}\in\mathscr{X}^\Lambda$. This inequality is clearly valid since

$$\boldsymbol{q}_t^{\boldsymbol{z}}(x)=\frac{\mathbf{P}_{t\cup\Lambda}(x\boldsymbol{z})}{\mathbf{P}_\Lambda(\boldsymbol{z})}=\frac{1}{\mathbf{P}_\Lambda(\boldsymbol{z})}\int\limits_{\{\overline{\boldsymbol{y}}\in\mathscr{X}^{t^{\mathrm{c}}}:\overline{\boldsymbol{y}}_\Lambda=\boldsymbol{z}\}} \mathbf{Q}_t^{\overline{\boldsymbol{y}}}(x)\ \mathbf{P}_{t^{\mathrm{c}}}(d\overline{\boldsymbol{y}}).$$

Taking this inequality into account, it remains to verify that

$$\lim_{\Lambda\uparrow\mathbb{Z}^\nu\setminus t}\ \sup_{\overline{\boldsymbol{x}}\in\mathscr{X}^{t^{\mathrm{c}}}} \left|\inf_{\overline{\boldsymbol{y}}\in\mathscr{X}^{t^{\mathrm{c}}}:\overline{\boldsymbol{y}}_\Lambda=\overline{\boldsymbol{x}}_\Lambda} \mathbf{Q}_t^{\overline{\boldsymbol{y}}}(x)-\mathbf{Q}_t^{\overline{\boldsymbol{x}}}(x)\right|=0$$

and

$$\lim_{\Lambda\uparrow\mathbb{Z}^\nu\setminus t}\ \sup_{\overline{\boldsymbol{x}}\in\mathscr{X}^{t^{\mathrm{c}}}} \left|\sup_{\overline{\boldsymbol{y}}\in\mathscr{X}^{t^{\mathrm{c}}}:\overline{\boldsymbol{y}}_\Lambda=\overline{\boldsymbol{x}}_\Lambda} \mathbf{Q}_t^{\overline{\boldsymbol{y}}}(x)-\mathbf{Q}_t^{\overline{\boldsymbol{x}}}(x)\right|=0$$

for all $t\in\mathbb{Z}^\nu$ and $x\in\mathscr{X}^t$. To show the first one we write

$$\sup_{\overline{\boldsymbol{x}}\in\mathscr{X}^{t^{\mathrm{c}}}} \left|\inf_{\overline{\boldsymbol{y}}\in\mathscr{X}^{t^{\mathrm{c}}}:\overline{\boldsymbol{y}}_\Lambda=\overline{\boldsymbol{x}}_\Lambda} \mathbf{Q}_t^{\overline{\boldsymbol{y}}}(x)-\mathbf{Q}_t^{\overline{\boldsymbol{x}}}(x)\right|\leqslant \sup_{\overline{\boldsymbol{x}}\in\mathscr{X}^{t^{\mathrm{c}}}}\ \sup_{\overline{\boldsymbol{y}}\in\mathscr{X}^{t^{\mathrm{c}}}:\overline{\boldsymbol{y}}_\Lambda=\overline{\boldsymbol{x}}_\Lambda} \left|\mathbf{Q}_t^{\overline{\boldsymbol{y}}}(x)-\mathbf{Q}_t^{\overline{\boldsymbol{x}}}(x)\right|$$

and use the quasilocality of $\boldsymbol{\mathcal{Q}}$. The second one is proved similarly. □

Roughly speaking, Criterion 7 asserts that aside from positivity consideration, Gibbs random fields are characterized by the uniform convergence of their one-point finite-conditional distribution (to the one-point conditional one), while only a weaker (almost sure) convergence is guaranteed for a general random field. In our opinion, this is perhaps the most comprehensible characterization of Gibbs random fields, on the basis of which the following purely probabilistic definition of Gibbs random field can be given.

Definition. — *A random filed* $\mathbf{P}$ *is called* Gibbs random field *if*

1) *for any* $\Lambda\in\mathscr{E}$ *and* $\boldsymbol{x}\in\mathscr{X}^\Lambda$ *one has* $\mathbf{P}_\Lambda(\boldsymbol{x})>0$,

2) *the limits*

$$\lim_{\Lambda\uparrow\mathbb{Z}^\nu\setminus t}\frac{\mathbf{P}_{t\cup\Lambda}(x\overline{\boldsymbol{x}}_\Lambda)}{\mathbf{P}_\Lambda(\overline{\boldsymbol{x}}_\Lambda)}\ ,\quad t\in\mathbb{Z}^\nu,\ x\in\mathscr{X}^t,\ \overline{\boldsymbol{x}}\in\mathscr{X}^{t^{\mathrm{c}}},$$

exist, are strictly positive, and the convergence is uniform with respect to $\overline{\boldsymbol{x}}$.

Taking this definition as the definition of Gibbs random field, the usual one turns into the following representation theorem.

Theorem. — *If* $\mathbf{P}$ *is a Gibbs random field, then* $\mathbf{P}$ *has a version of conditional distribution which can be expressed by Gibbs formulae* (3) *and* (4) *with some uniformly convergent potential.*

Conversely, if a random field $\mathbf{P}$ *has a version of conditional distribution which can be expressed by Gibbs formulae* (3) *and* (4) *with some uniformly convergent potential, then* $\mathbf{P}$ *is a Gibbs random field.*

in terms of one-point finite-conditional distribution II

At first sight, the above presented Criterion 7 deals only with one-point finite-conditional distribution. However, in fact it imposes conditions equally on its limit, that is, on one-point conditional distribution. The following and last criterion really deals only with one-point finite-conditional distribution. Before formulating it, let as agree that in the sequel when we use the notation $\boldsymbol{x}_T$ we presume that only configurations $\boldsymbol{x}$ such that $\mathfrak{S}(\boldsymbol{x}) \supset T$ are considered.

Criterion 8. — *A random field is a Gibbs random field if and only if it is strictly positive, its one-point finite-conditional distribution* $\big\{\boldsymbol{q}_t^{\widetilde{\boldsymbol{x}}},\quad t \in \mathbb{Z}^\nu$ *and* $\widetilde{\boldsymbol{x}} \in \widetilde{\mathscr{X}^t}\big\}$ *is uniformly nonnull and one of the following equivalent conditions holds:*

(C) *for any* $t \in \mathbb{Z}^\nu$ *and* $x \in \mathscr{X}^t$ *one has*

$$\lim_{\Lambda\uparrow\mathbb{Z}^\nu\setminus t}\ \sup_{\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{y}}\in\widetilde{\mathscr{X}^t}:\,\widetilde{\boldsymbol{x}}_\Lambda=\widetilde{\boldsymbol{y}}_\Lambda}\ \left|\boldsymbol{q}_t^{\widetilde{\boldsymbol{x}}}(x)-\boldsymbol{q}_t^{\widetilde{\boldsymbol{y}}}(x)\right| = 0,$$

(D) *for any* $t \in \mathbb{Z}^\nu$ *and* $x \in \mathscr{X}^t$ *one has*

$$\lim_{\Lambda\uparrow\mathbb{Z}^\nu\setminus t}\ \sup_{J\in\mathscr{E}^*(t^c)}\ \sup_{\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{y}}\in\mathscr{X}^J:\,\widetilde{\boldsymbol{x}}_\Lambda=\widetilde{\boldsymbol{y}}_\Lambda}\ \left|\boldsymbol{q}_t^{\widetilde{\boldsymbol{x}}}(x)-\boldsymbol{q}_t^{\widetilde{\boldsymbol{y}}}(x)\right| = 0,$$

(E) *for any* $t \in \mathbb{Z}^\nu$ *and* $x \in \mathscr{X}^t$ *one has*

$$\lim_{\Lambda\uparrow\mathbb{Z}^\nu\setminus t}\ \sup_{\widetilde{\boldsymbol{x}}\in\widetilde{\mathscr{X}^t}}\ \left|\boldsymbol{q}_t^{\widetilde{\boldsymbol{x}}}(x)-\boldsymbol{q}_t^{\widetilde{\boldsymbol{x}}_\Lambda}(x)\right| = 0.$$

Proof. First we concentrate on the condition (E). Clearly

$$\sup_{\widetilde{\boldsymbol{x}}\in\widetilde{\mathscr{X}^t}}\ \left|\boldsymbol{q}_t^{\widetilde{\boldsymbol{x}}}(x)-\boldsymbol{q}_t^{\widetilde{\boldsymbol{x}}_\Lambda}(x)\right| = \sup_{I\in\mathscr{E}(t^c):\,I\supset\Lambda}\ \sup_{\overline{\boldsymbol{x}}\in\mathscr{X}^{t^c}}\ \left|\boldsymbol{q}_t^{\overline{\boldsymbol{x}}_I}(x)-\boldsymbol{q}_t^{\overline{\boldsymbol{x}}_\Lambda}(x)\right|,$$

and so the condition (E) is nothing but the Cauchy condition for the existence of the uniform limits considered in Criterion 7. The sufficiency now clearly follows from Criterion 7 since the Cauchy principle yields the existence of the uniform limits, and the uniform nonnullness of one-point finite-conditional distribution guarantees their strict positivity. The necessity also follows from Criterion 7 since the condition (E) is ensured by the Cauchy principle, and the uniform nonnullness of one-point finite-conditional distribution can be easily obtained from (9) and the condition (A) (use the first inequality of (9) and the uniform nonnullness of limits considered in the condition (A)).

It remains to check the equivalence of the conditions (C), (D) and (E). The implications (C)⇒(D) and (C)⇒(E) are trivial since

$$\sup_{J\in\mathscr{E}^*(t^c)}\ \sup_{\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{y}}\in\mathscr{X}^J:\,\widetilde{\boldsymbol{x}}_\Lambda=\widetilde{\boldsymbol{y}}_\Lambda}\ \left|\boldsymbol{q}_t^{\widetilde{\boldsymbol{x}}}(x)-\boldsymbol{q}_t^{\widetilde{\boldsymbol{y}}}(x)\right| \leqslant \sup_{\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{y}}\in\widetilde{\mathscr{X}^t}:\,\widetilde{\boldsymbol{x}}_\Lambda=\widetilde{\boldsymbol{y}}_\Lambda}\ \left|\boldsymbol{q}_t^{\widetilde{\boldsymbol{x}}}(x)-\boldsymbol{q}_t^{\widetilde{\boldsymbol{y}}}(x)\right|$$

and

$$\sup_{\widetilde{\boldsymbol{x}}\in\widetilde{\mathscr{X}^t}}\ \left|\boldsymbol{q}_t^{\widetilde{\boldsymbol{x}}}(x)-\boldsymbol{q}_t^{\widetilde{\boldsymbol{x}}_\Lambda}(x)\right| \leqslant \sup_{\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{y}}\in\widetilde{\mathscr{X}^t}:\,\widetilde{\boldsymbol{x}}_\Lambda=\widetilde{\boldsymbol{y}}_\Lambda}\ \left|\boldsymbol{q}_t^{\widetilde{\boldsymbol{x}}}(x)-\boldsymbol{q}_t^{\widetilde{\boldsymbol{y}}}(x)\right|.$$

$$\sup_{\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{y}}\in\widetilde{\mathscr{X}^t}\,:\,\widetilde{\boldsymbol{x}}_\Lambda=\widetilde{\boldsymbol{y}}_\Lambda} \left|\boldsymbol{q}_t^{\widetilde{\boldsymbol{x}}}(x)-\boldsymbol{q}_t^{\widetilde{\boldsymbol{y}}}(x)\right| \leqslant 2 \sup_{\widetilde{\boldsymbol{x}}\in\widetilde{\mathscr{X}^t}} \left|\boldsymbol{q}_t^{\widetilde{\boldsymbol{x}}}(x)-\boldsymbol{q}_t^{\widetilde{\boldsymbol{x}}_\Lambda}(x)\right|$$

yields the implications (E)⇒(C). To prove the last implication (D)⇒(C), we need the following lemma.

Lemma. — *Let* $\left\{\boldsymbol{q}_I^{\widetilde{\boldsymbol{x}}},\quad I\in\mathscr{E} \text{ and } \widetilde{\boldsymbol{x}}\in\widetilde{\mathscr{X}^I}\right\}$ *be the finite-conditional distribution of some strictly positive random field. Then the set*

$$A=\left\{\overline{\boldsymbol{x}}\in\mathscr{X}^{\mathbb{Z}^\nu}\;:\;\lim_{\Lambda\uparrow\mathbb{Z}^\nu\setminus I}\boldsymbol{q}_I^{\overline{\boldsymbol{x}}_\Lambda}(x)\text{ exists for every } I\in\mathscr{E}\text{ and } x\in\mathscr{X}^I\right\}$$

is of probability 1 *and possesses the following property: if* $\overline{\boldsymbol{x}}\in A$ *then* $\boldsymbol{z}\overline{\boldsymbol{x}}_{J^{\mathrm{c}}}\in A$ *for all* $J\in\mathscr{E}$ *and* $\boldsymbol{z}\in\mathscr{X}^J$.

Proof. Since the set A is a countable intersection of sets of probability 1, it is also of probability 1. It remains to show that if $\overline{\boldsymbol{x}}\in A$ then $\overline{\boldsymbol{y}}=z\overline{\boldsymbol{x}}_{t^{\mathrm{c}}}\in A$ for all $t\in\mathbb{Z}^\nu$ and $z\in\mathscr{X}^t$, that is, $\lim\limits_{\Lambda\uparrow\mathbb{Z}^\nu\setminus I}\boldsymbol{q}_I^{\overline{\boldsymbol{y}}_\Lambda}(\boldsymbol{x})$ exists for every $I\in\mathscr{E}$ and $\boldsymbol{x}\in\mathscr{X}^I$. This is trivial if $t\in I$ (since in this case $\overline{\boldsymbol{y}}_\Lambda=\overline{\boldsymbol{x}}_\Lambda$) and clearly follows from the relation

$$\boldsymbol{q}_I^{\overline{\boldsymbol{y}}_\Lambda}(\boldsymbol{x})=\boldsymbol{q}_I^{z\overline{\boldsymbol{x}}_{\Lambda\setminus t}}(\boldsymbol{x})=\frac{\boldsymbol{q}_{t\cup I}^{\overline{\boldsymbol{x}}_{\Lambda\setminus t}}(z\boldsymbol{x})}{\left(\boldsymbol{q}_{t\cup I}^{\overline{\boldsymbol{x}}_{\Lambda\setminus t}}\right)_t(z)},\quad \Lambda\ni t,$$

otherwise. ◁

Returning to the proof of the implication (D)⇒(C), let us fix some $t\in\mathbb{Z}^\nu$ and $x\in\mathscr{X}^t$, denote

$$f(\Lambda)=\sup_{J\in\mathscr{E}^*(t^{\mathrm{c}})}\;\sup_{\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{y}}\in\mathscr{X}^J\,:\,\widetilde{\boldsymbol{x}}_\Lambda=\widetilde{\boldsymbol{y}}_\Lambda}\left|\boldsymbol{q}_t^{\widetilde{\boldsymbol{x}}}(x)-\boldsymbol{q}_t^{\widetilde{\boldsymbol{y}}}(x)\right|,$$

and for any $\varepsilon>0$ choose (according to the condition (D)) some $\Lambda_\varepsilon\in\mathscr{E}$ such that $|f(\Lambda)|<\varepsilon$ for all $\Lambda\in\mathscr{E}$, $\Lambda\supset\Lambda_\varepsilon$.

First, we will show that $\lim\limits_{\Lambda\uparrow\mathbb{Z}^\nu\setminus t}\boldsymbol{q}_t^{\overline{\boldsymbol{x}}_\Lambda}(x)$ exists for every $\overline{\boldsymbol{x}}\in\mathscr{X}^{t^{\mathrm{c}}}$. Let us take some $\overline{\boldsymbol{x}}^\circ\in A$ (according to the lemma, the set A is of probability 1 and so is not empty) and consider $\overline{\boldsymbol{y}}=\overline{\boldsymbol{x}}_{\Lambda_\varepsilon}\overline{\boldsymbol{x}}^\circ_{\Lambda_\varepsilon^{\mathrm{c}}}\in A$. So, we can find some $\Lambda'_\varepsilon\in\mathscr{E}$ such that $\left|\boldsymbol{q}_t^{\overline{\boldsymbol{y}}_I}(x)-\boldsymbol{q}_t^{\overline{\boldsymbol{y}}_J}(x)\right|<\varepsilon$ for all $I,J\in\mathscr{E}$, $I\supset\Lambda'_\varepsilon$, $J\supset\Lambda'_\varepsilon$. Thus, for all $I,J\in\mathscr{E}$ such that $I\supset\Lambda_\varepsilon\cup\Lambda'_\varepsilon$ and $J\supset\Lambda_\varepsilon\cup\Lambda'_\varepsilon$ we can write

$$\begin{aligned}\left|\boldsymbol{q}_t^{\overline{\boldsymbol{x}}_I}(x)-\boldsymbol{q}_t^{\overline{\boldsymbol{x}}_J}(x)\right| &\leqslant \left|\boldsymbol{q}_t^{\overline{\boldsymbol{x}}_I}(x)-\boldsymbol{q}_t^{\overline{\boldsymbol{y}}_I}(x)\right|+\left|\boldsymbol{q}_t^{\overline{\boldsymbol{y}}_I}(x)-\boldsymbol{q}_t^{\overline{\boldsymbol{y}}_J}(x)\right|+\left|\boldsymbol{q}_t^{\overline{\boldsymbol{y}}_J}(x)-\boldsymbol{q}_t^{\overline{\boldsymbol{x}}_J}(x)\right|\\ &< f(\Lambda_\varepsilon)+\varepsilon+f(\Lambda_\varepsilon)<3\varepsilon,\end{aligned}$$

and hence $\lim\limits_{\Lambda\uparrow\mathbb{Z}^\nu\setminus t}\boldsymbol{q}_t^{\overline{\boldsymbol{x}}_\Lambda}(x)$ exists according to Cauchy principle.

Further, for every $\overline{\boldsymbol{x}}\in\mathscr{X}^{t^{\mathrm{c}}}$ consider the set $V(\overline{\boldsymbol{x}})=\left\{\overline{\boldsymbol{y}}\in\mathscr{X}^{t^{\mathrm{c}}}\;:\;\overline{\boldsymbol{y}}_{\Lambda_\varepsilon}=\overline{\boldsymbol{x}}_{\Lambda_\varepsilon}\right\}$. Clearly these sets are either mutually disjoint or coinciding, and there is only a finite

exists a finite collection $\overline{\boldsymbol{x}}^1,\dots,\overline{\boldsymbol{x}}^k \in \mathscr{X}^{t^c}$ such that

$$\mathscr{X}^{t^c} = \bigcup_{i=1}^{k} V(\overline{\boldsymbol{x}}^i).$$

(This fact equally follows from the compactness of $\mathscr{X}^{t^c}$.) So, using Cauchy principle we can find some $\Lambda''_\varepsilon \in \mathscr{E}$ such that $\left|\boldsymbol{q}_t^{\overline{\boldsymbol{x}}^i_I}(x) - \boldsymbol{q}_t^{\overline{\boldsymbol{x}}^i_J}(x)\right| < \varepsilon$ for all $i = 1,\dots,k$ and all $I, J \in \mathscr{E}$, $I \supset \Lambda''_\varepsilon$, $J \supset \Lambda''_\varepsilon$.

Now, let the set $\Lambda \in \mathscr{E}$ be such that $\Lambda \supset \Lambda_\varepsilon \cup \Lambda''_\varepsilon$, the sets $I, J \in \mathscr{E}$ be such that $I \supset \Lambda$ and $J \supset \Lambda$, and the configurations $\widetilde{\boldsymbol{x}} \in \mathscr{X}^I$ and $\widetilde{\boldsymbol{y}} \in \mathscr{X}^J$ be such that $\widetilde{\boldsymbol{x}}_\Lambda = \widetilde{\boldsymbol{y}}_\Lambda$. Clearly, we can find some $i \in \{1,\dots,k\}$ such that $\overline{\boldsymbol{x}}^i_{\Lambda_\varepsilon} = \widetilde{\boldsymbol{x}}_{\Lambda_\varepsilon} = \widetilde{\boldsymbol{y}}_{\Lambda_\varepsilon}$, and thus we may write

$$\begin{aligned}\left|\boldsymbol{q}_t^{\widetilde{\boldsymbol{x}}}(x) - \boldsymbol{q}_t^{\widetilde{\boldsymbol{y}}}(x)\right| &\leqslant \left|\boldsymbol{q}_t^{\widetilde{\boldsymbol{x}}}(x) - \boldsymbol{q}_t^{\overline{\boldsymbol{x}}^i_I}(x)\right| + \left|\boldsymbol{q}_t^{\overline{\boldsymbol{x}}^i_I}(x) - \boldsymbol{q}_t^{\overline{\boldsymbol{x}}^i_J}(x)\right| + \left|\boldsymbol{q}_t^{\overline{\boldsymbol{x}}^i_J}(x) - \boldsymbol{q}_t^{\widetilde{\boldsymbol{y}}}(x)\right| \\ &< f(\Lambda_\varepsilon) + \varepsilon + f(\Lambda_\varepsilon) < 3\varepsilon\end{aligned}$$

which shows that the condition (C) holds. □

In conclusion let us note that the analogues of Criteria 7 and 8 formulated in terms of all the finite-conditional distribution are of course valid. Concerning the first one, we would like to mention that its necessity statement was as a matter of fact contained in the proof of Lemma 1 of Sullivan [17], whose argument we follow while proving Criterion 7. As to the second one, let us mention that the part utilizing the analogue of the condition (D) can be deduced from Theorems 1 and 2 of Kozlov [13]. It should be pointed out that the author does not provide the proof of the sufficiency statement of Theorem 2 (leaving it, as he says, to the reader). However, our considerations show that the proof of this statement is neither intuitive, nor technically simple. Moreover, the validity of the statement seems dubitable in the settings of Kozlov [13] where the state space is not supposed to be finite or even compact.

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